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Interacting columns: generating functions and scaling exponents

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Abstract. A model of self-interacting columns, related to models of partially directed walks and to histogram polygons, is considered. The generating function of this model is found in terms of q-deformed Bessel functions using a functional recursion scheme. A transition, related to a deflation–inflation transition seen in staircase polygons, or to a rough–smooth transition in a solid-on-solid model, is found, and its scaling exponents are found in the context of a tricritical scaling analysis. If the columns are also interacting with the horizontal axis, then the inflated or smooth phase is also found to undergo an adsorption transition. A special point exists where the model is critical with respect to both an adsorption transition and a deflation–inflation transition.

1. Introduction

The generating functions of geometric cluster models have received considerable attention in the literature (Owczarek *et al* 1993, Brak *et al* 1993). Of particular interest are the generating functions in these models, and finding such generating functions is an activity also of interest in combinatorial mathematics (Pólya 1969, Delest 1988). From the point of view in statistical mechanics, these models have *phase diagrams* which include multicritical points, and there they serve as examples of models with interesting thermodynamic behaviour. Of particular interest here are models of directed walks (Brak *et al* 1992, Whittington 1998) and convex models of polygons in the square lattice (Pólya 1969, Bousquet-Mélou 1992, 1994, Brak *et al* 1994, Prellberg and Brak 1995).

There are a variety of methods for the determination of generating functions in these directed or convex models, including the Temperley method (Temperley 1956) involving recursions (Privman and Švrakić 1988), or functional recursions (Prellberg and Brak 1995). The usual observation in all of these models is that the one-variable generating function is algebraic, but a two- or more variable generating function involves basic special functions (including q-deformations of the factorial, the exponential and Bessel functions). For example, the connection between the q-exponential and the two-variable area–perimeter generating function of partition polygons (Ferrers or Young diagrams) is well known; this is often discussed in standard texts of enumerative combinatorics (see, for example, Stanley 1986).

A *column* of height n is a walk in the square lattice (XY-plane) with starting point on the X-axis, and which steps n times in the Y-direction, then one step in the X-direction followed by n steps in the $-Y$-direction to terminate on the X-axis. The perimeter of the column consists of $2n+1$ edges, and it encloses an area of n unit squares. Two columns are adjacent if they are separated by one step in the X-direction; a column and a collection of adjacent columns are illustrated in figure 1. Adjacent columns interact via *contacts* which are pairs of vertices (one

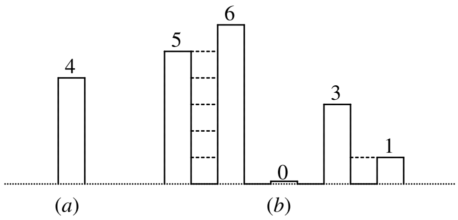

Figure 1. (*a*) A column of height 4. (*b*) A set of interacting columns. Broken lines indicate contacts between adjacent columns. The column of height 0 is a *visit*.

in each column) that are adjacent in the lattice. A model of columns can also interact with the X-axis if those of height zero are said to be *visits*. An interacting model of a collection of adjacent columns is defined by introducing activities conjugate to contacts (y) and to visits (z). In addition, x will be an activity conjugate to the total area of the collection of columns (note that x is also conjugate to pairs of vertical edges in a collection of columns). A collection of columns is said to be *connected* if every pair of columns is in a sequence $\{\gamma_i\}$ of columns such that γ_i and γ_{i+1} are adjacent. A key parameter in the model here is an activity η conjugate to the number of columns in a connected collection of columns.

I show in section 2 that the generating function of a collection of connected columns is, in fact, also the generating function of histogram polygons or bar-graph polygons (after a suitable change of variables is made) (Prellberg and Brak 1995). The (upper) perimeter of a histogram polygon is also a partially directed walk (which may only step East, North or South), so the model in this paper is related to a model of partially directed walks interacting with the X-axis (Whittington 1998).

Define $c_n(k, v)$ to be the number of connected columns with total area n, k contacts between adjacent columns and v visits (columns of height zero). The canonical partition function of this model is

$$c_n(y, z) = \sum_{k,v} c_n(k, v)\, y^k z^v. \tag{1.1}$$

The limiting free energy (per unit area) is defined by

$$\mathcal{F}(y, z) = \lim_{n\to\infty} \frac{1}{n} \log c_n(y, z). \tag{1.2}$$

The singularities in the limiting free energy correspond to phase transitions, and the singularity diagram of $\mathcal{F}(y, z)$ will be the phase diagram of the model. There are several phases possible in this model. Increasing the value of y will presumably take the model through a collapse transition from a rough phase to a smooth phase (I shall refer to this as a rough–smooth transition). This should be similar to a roughening transition in, for example, an SOS model (Owczarek and Prellberg 1993). Increasing z should take the model through an adsorption transition, similar to that observed in directed models of directed and partially directed walks (Whittington 1998, Janse van Rensburg 1999).

The generating function of this model is

$$G(x, y, z) = \sum_{n=0}^{\infty} c_n(y, z)\, x^n. \tag{1.3}$$

The existence of the limit in equation (1.2) follows from a supermultiplicative relationship involving $c_n(y, z)$. In particular, consider two collections of columns, the first with k_1 contacts,

v_1 visits, and area n, and the second with $k - k_1$ contacts, $v - v_1$ visits and area m. Translate the second collection of columns until its vertex with least X-coordinate is separated from that vertex with greatest X-coordinate by one column of height one. This creates two extra contacts, and one new column of height one, in a new concatenated collection of columns with $k + 2$ contacts, v visits and area $n + m + 1$. Since there are $c_n(k_1, v_1)$ choices for the first set of columns and $c_m(k - k_1, v - v_1)$ choices for the second set of columns,

$$\sum_{k_1=0}^{k} \sum_{v_1=0}^{v} c_n(k_1, v_1) c_m(k - k_1, v - v_1) \leqslant c_{n+m+1}(k + 2, v). \tag{1.4}$$

Multiply this by $y^k z^v$ and sum over k and v to find that

$$c_n(y, z) c_m(y, z) \leqslant y^{-2} c_{n+m+1}(y, z). \tag{1.5}$$

Finally, since† $c_n(k, v) \leqslant 2^n$, existence of the limit in equation (1.2) follows from a standard theorem on subadditive functions (Hille 1948). Naturally, if the radius of convergence of $G(x, y, z)$ is $x_c(y, z)$, then

$$\mathcal{F}(y, z) = -\log x_c(y, z). \tag{1.6}$$

If an activity η conjugate to the number of columns is introduced, then similar arguments demonstrate the existence of $\mathcal{F}_\eta(y, z) = -\log x_c(y, z)$, and where $x_c(y, z)$ is now the radius of convergence of the generating function

$$G(x, y, z) = \sum_{n=0}^{\infty} c_n(y, z, \eta)\, x^n. \tag{1.7}$$

Naturally, $c_n(y, z, \eta)$ is the partition function of a model of collections of columns with activities η, y and z.

The central object of interest will be the generating function, and I shall derive it by solving a functional relation involving $G(x, y, z)$. Critical points in this type of model are non-analyticities in $\mathcal{F}(y, z)$, and thus the analyticity of $x_c(y, z)$ is important. Points on $x_c(y, z)$ where the nature of the singularity in $G(x, y, z)$ changes are also points of special interest. In the most interesting cases a curve of essential singularities in $G(x, y, z)$ meets a curve of simpler singularities (branch points or poles); these points are tricritical points in the singularity diagram, and in their vicinity the generating function is thought to be subject to tricritical scaling (see, for example, Brak *et al* 1993, Janse van Rensburg 2000).

The situation is slightly simpler if a two-variable generating function is considered instead. Let $g(x, z)$ be a two-variable generating function (possibly involving q-deformed special functions) of a model whose phase diagram includes a tricritical point. In the vicinity of the tricritical point (x_c, z_c) it is expected that $g(x, z)$ will exhibit tricritical scaling. In particular, the asymptotic behaviour of $g(x, z)$ should be

$$g(x, z) \sim (x_c - x)^{2-\alpha_t} f((z_c - z)(x_c - x)^{-\phi}) \tag{1.8}$$

and given a certain set of assumptions, this is a uniform asymptotic description of the generating function (see Lawrie and Sarlbach 1984, Brak *et al* 1993). The function $f(x)$ in equation (1.8) is a function with the properties that $f(\infty)$ is a constant, and $f(x) \sim x^u$ if x is small. The tricritical exponents $2 - \alpha_t$ and ϕ describe the universal behaviour in the model, ϕ is called the crossover exponent. It is usually assumed that $2 - \alpha_t \geqslant 0$ so that $g(x, z)$ is convergent if

† To see this, note that sets of connected columns of area n are in one-to-one correspondence with the number of non-negative integer solutions of $n = a_1 + a_2 + \cdots + a_m$.

$x \to x_c^-$. In some cases the generating function is divergent as $x \to x_c^-$, and in that case the usual scaling assumption would be

$$g(x, z) \sim (x_c - x)^{-\gamma_t} f((z_c - z)(x_c - x)^{-\phi}) \tag{1.9}$$

with the exponent $-\gamma_t < 0$ replacing $2 - \alpha_t$ (Brak *et al* 1993).

The asymptotic behaviour of $g(x, z)$ if (x_c, z_c) is approached along the x-direction (with $z = z_c$ fixed) is described by the exponent $2 - \alpha_t$ (or γ_t); but a different exponent is expected if the approach is along the curve $x_c(z)$;

$$g(x, z) \sim (z_c - z)^{2-\alpha_u} \tag{1.10}$$

and standard arguments in the theory of tricritical scaling indicate that

$$\frac{2 - \alpha_t}{2 - \alpha_u} = \phi. \tag{1.11}$$

In some models the exponent $-\gamma_u$ is appropriate (instead of $2 - \alpha_u$), and then this relation is $\gamma_t/\gamma_u = \phi$.

2. Collapsing columns

2.1. Columns with a contact activity

It will be convenient to initially ignore visits in a collection of columns. Thus, in this section I shall only consider collections of columns where each column has a height of at least one. Let the (two-variable) generating function of this model be $G_1(x, y)$, where x is the area-generating variable (or x^2 is the perimeter-generating variable), and y is the contact-generating variable. To find $G_1(x, y)$, consider a collection of columns, and increase the height of each by one (this process will be called *inflation*). This is illustrated in figure 2. An inflated collection of columns will have each column of height at least two.

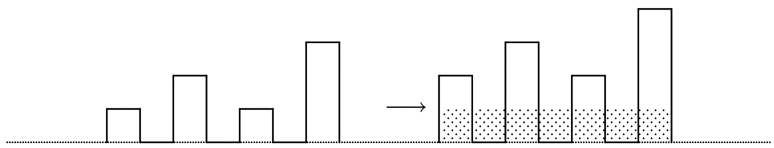

Figure 2. Inflating a collection of columns raises the height of each column by one.

It will be useful to introduce an activity η conjugate to the number of columns in this model. In other words, let $G_1(x, y, \eta)$ be the generating function of collections of columns with no visits, where y is the contact-generating variable, η is the column-generating variable and x is the area- or perimeter-generating variable. Since each inflated collection of columns can be obtained by inflating a collection of columns counted by $G_1(x, y, \eta)$, one only has to observe that if there are N columns in a collection, then inflation will produce $N - 1$ new contacts, and increase the area by N, to write down the generating function for inflated collections of columns. Each factor of η gives rise to a factor of x, and a factor y, in the inflation (but this overcounts the new factors of y by one, since there is one less new contact than columns). Thus, the inflated collections of columns have a generating function $y^{-1}G_1(x, y, xy\eta)$.

Each collection of columns without visits can now be classified in the following way. It is either an inflated collection of columns (counted by $y^{-1}G_1(x, y, xy\eta)$), or it has a column of height one. If it has a column of height one, then it is either a single column of height one, or its

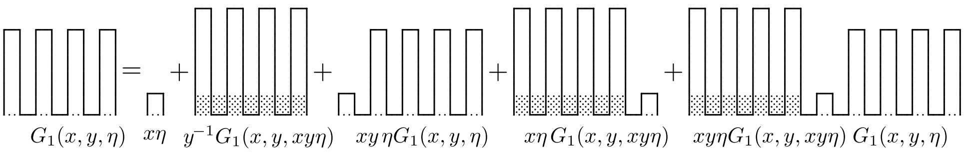

Figure 3. Every collection of columns can be classified into one of the classes indicated by this schematic drawing.

first column has height one, or it consists of an inflated column, followed by a column of height one (which is either the last column, or is followed by more columns). This classification is illustrated in figure 3.

The result is the following functional equation for $G_1(x, y, \eta)$ (where I suppress the x and y arguments: $G_1(\eta) \equiv G_1(x, y, \eta)$):

$$\begin{aligned} G_1(\eta) &= x\eta + y^{-1}G_1(xy\eta) + xy\eta G_1(\eta) + x\eta G_1(xy\eta) + xy\eta G_1(\eta)G_1(xy\eta) \\ &= x\eta + x^2\eta(1 + yn) + x^3\eta(1 + y\eta)^2 + \mathrm{O}(x^4) \end{aligned} \tag{2.1}$$

and this may be viewed as a power series in x with polynomial coefficients in y and η. This can be simplified by defining

$$q = xy \tag{2.2}$$

to obtain

$$G_1(\eta) = \frac{q\eta}{y} + q\eta G_1(\eta) + \frac{1}{y}(1 + q\eta)G_1(q\eta) + q\eta G_1(\eta)G_1(q\eta). \tag{2.3}$$

This is a nonlinear equation for $G_1(\eta)$, and it can be written as

$$G_1(\eta)G_1(q\eta) + \frac{1 + q\eta}{yq\eta}G_1(q\eta) + \left(1 - \frac{1}{q\eta}\right)G_1(\eta) + \frac{1}{y} = 0 \tag{2.4}$$

where $G_1(\eta)$ may be considered a power series in η with rational coefficients in y and q. It is possible to solve explicitly for $G_1(\eta)$ from this equation if $q = 1$: a direct calculation gives

$$G_1(\eta)|_{q=1} = (1 - \eta - x - x\eta - \sqrt{(1 - x)((1 - \eta^2) - x(1 + \eta^2))})/2\eta. \tag{2.5}$$

By expanding this it can be checked that the first couple of terms in equation (2.1) are generated. Note that there are non-analyticities in $G_1(\eta)|_{q=1}$ at $x = 1$ and at $x = (1 - \eta)^2/(1 + \eta)^2$. Now $G_1(\eta)|_{q=1}$ is analytic on the line $q = 1$ if $x > 1$ and if $x < (1 - \eta)^2/(1 + \eta)^2$, and the square-root singularities in equation (2.5) at these points suggest that they are candidates as tricritical points with $2 - \alpha_u = \frac{1}{2}$ (see equation (1.10)).

A full solution for $G_1(\eta)$ can be found by solving the functional recursion in equation (2.4) (Prellberg and Brak 1995). The starting point is to linearize equation (2.4) by substituting the following ansatz:

$$G_1(\eta) = \frac{B}{\eta}\frac{H(q\eta)}{H(\eta)} - \frac{1 + q\eta}{yq\eta}. \tag{2.6}$$

After simplification, this gives

$$yqB^2H(q^2\eta) + (yq^2\eta - 1 - q^2\eta - yq)B\,H(q\eta) + H(\eta) = 0. \tag{2.7}$$

Define the following quantities:

$$\alpha_0 = 1/Bq^2(y - 1) \qquad \alpha_1 = -(1 + yq)/q^2(y - 1) \qquad \alpha_2 = Byq/q^2(y - 1). \tag{2.8}$$

Then equation (2.7) is

$$\alpha_2 H(q^2\eta) + \alpha_1 H(q\eta) + \alpha_0 H(\eta) + \eta H(q\eta) = 0 \tag{2.9}$$

and if

$$B = \frac{1 + yq \pm (1 - yq)}{2yq} \tag{2.10}$$

then

$$\alpha_0 + \alpha_1 + \alpha_2 = 0 \tag{2.11}$$

in equation (2.9). The sign in B (equation (2.10)) must be chosen to give the correct expansion for $G_1(\eta)$.

It remains to find $H(\eta)$ in equation (2.9). It is the case that a functional recursion of that type, subject to the constraint in equation (2.11) has the solution (Prellberg and Brak 1995)

$$H(\eta) = \sum_{n=0}^{\infty} \frac{(-\eta)^n q^{\binom{n}{2}}}{\prod_{m=1}^{n} \Lambda(q^m)} \tag{2.12}$$

where

$$\Lambda(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2. \tag{2.13}$$

Since $\alpha_0 + \alpha_1 + \alpha_2 = 0$, one root of $\Lambda(t)$ is $t = 1$, and the second one is α_0/α_2, in particular,

$$\Lambda(t) = \alpha_0(1 - t)\left(1 - \frac{\alpha_2}{\alpha_0} t\right). \tag{2.14}$$

Define the q-product by

$$(t; q)_n = (1 - t)(1 - tq)(1 - tq^2) \ldots (1 - tq^{n-1}) = \prod_{m=0}^{n-1} (1 - tq^m) \tag{2.15}$$

then

$$\prod_{m=1}^{n} \Lambda(q^m) = \alpha_0^n (q; q)_n (\alpha_2 q/\alpha_0; q)_n. \tag{2.16}$$

It follows that $\alpha_2/\alpha_0 = B^2 yq$, and thus

$$H(\eta) = \sum_{n=0}^{\infty} \frac{(-Bq^2(y-1)\eta)^n q^{\binom{n}{2}}}{(q; q)_n (B^2 yq^2; q)_n}. \tag{2.17}$$

With these expressions, the generating function $G_1(x, y, \eta)$ is completely determined. $H(\eta)$ is also a q-deformed Bessel function; and if the q-Bessel function $J(x, y, q)$ is defined by

$$J(x, y, q) = \sum_{n=0}^{\infty} \frac{(-x)^n q^{\binom{n}{2}}}{(q; q)_n (y; q)_n} \tag{2.18}$$

then

$$H(\eta) = J(Bq^2(y-1)\eta, B^2 yq^2; q). \tag{2.19}$$

Note that B is also a function of q, and that the generating function involves a ratio of q-deformed Bessel functions;

$$G_1(x, y, \eta) = \frac{B}{\eta} \frac{J(B(y-1)q^3\eta, B^2 yq^2, q)}{J(B(y-1)q^2\eta, B^2 yq^2, q)} - \frac{1 + q\eta}{yq\eta}. \tag{2.20}$$

The choice for B can now be made. Note that the last term in equation (2.20) contains a term to order $O(y^{-1}\eta^{-1})$. B must cancel this term, and only the choice $B = 1/yq$ (corresponding to the plus sign in equation (2.10)) achieves this. Thus, the complete solution is

$$G_1(x, y, \eta) = \frac{1}{yq\eta}\left(\frac{J((1-1/y)q^2\eta, 1/y, q)}{J((1-1/y)q\eta, 1/y, q)} - 1 - q\eta\right) = \frac{1}{yq\eta}\left(\frac{H(q\eta)}{H(\eta)} - 1 - q\eta\right) \tag{2.21}$$

where $q = xy$. This solution is similar to the area–perimeter-generating function of staircase polygons and other convex models of polygons considered previously (Brak *et al* 1992, 1994, Prellberg 1995). The functional equation for $G_1(\eta)$ (equation (2.1)) can, in fact, be written as

$$G_1(\eta) = G_1(xy\eta)/y + (1/y + G_1(\eta))xy\eta(1 + G_1(xy\eta)) \tag{2.22}$$

and if first $q = xy$ and then $\eta \to x$ and $y \to 1/y$, then the functional recursion for bar-graph polygons, or histogram polygons, in an area–perimeter functional recursion, is obtained (see equation (3.11) in Prellberg and Brak (1995)).

2.2. *The singularity structure of the generating function of columns with a contact activity*

In analogy with staircase polygons (which have a similar, but not identical) generating function), it may be expected that there is a line of essential singularities in $G_1(x, y, \eta)$ at $q = xy = 1$. In addition, if $G_1(x, y, \eta)$ is restricted to the curve $xy = 1$ in the xy-plane, then it is analytic along at least part of this curve, as may be seen from equation (2.5). If $y = 0$, then $G_1(x, 0, \eta) = x\eta/(1-x)$, and thus the point $(y, x) = (0, 1)$ is a simple pole in the generating function. On the other hand, if $y = 1$, then the generating function is given by

$$G_1(x, 1, \eta) = \sum_{n=1}^{\infty}\sum_{k=1}^{n} A_n(k)\,\eta^k x^n \tag{2.23}$$

where $A_n(k)$ is the number of collections of columns with k columns and total area n. Naturally,

$$A_n(k) = \binom{n-1}{k-1} \tag{2.24}$$

so that direct calculation gives

$$G_1(x, 1, \eta) = \frac{x\eta}{1 - x(1+\eta)} = x\eta + x^2\eta + x^2\eta^2 + x^3\eta + \cdots. \tag{2.25}$$

Direct substitution into equation (2.4) (with $y = 1$) shows that this is indeed a solution of equation (2.1). There is a simple pole in $G_1(x, 1, \eta)$ at $x = 1/(1+\eta) < 1$, and so one may guess that the radius of convergence of $G_1(x, y, \eta)$, $x_c(y)$, is not determined by the curve $x = y^{-1}$, but by a curve $x_c(y)$ of simple poles lying below the curve $x = y^{-1}$ in the xy-plane. Singularities of $G_1(x, y, \eta)$ below $x = y^{-1}$ should be due to the roots of $H(\eta)$ in equation (2.21)†. Observe that $H(\eta)$ decreases from 1 to $-\infty$ as x increases from 0 to $\min(1, y^{-1})$. Since $H(\eta)$ is continuous, there is an $x_c(y)$ where $H(\eta) = 0$ (by the intermediate value theorem). Moreover, $\frac{\partial H(\eta)}{\partial x} < 0$, so this pole is simple. This demonstrates the existence of a simple pole in $G_1(x, y, \eta)$ at $x_c(y)$ for each $y \leqslant 1$. I assume that these simple poles will all lie on a curve $x_c(y)$. The curve $x_c(y)$ should be non-increasing with y and it should meet $xy = 1$ at a tricritical point; the obvious candidate is a critical point along $xy = 1$, which can

† Observe that $H(\eta)$ in equation (2.19) is absolutely convergent for $q < 1$ and $x < 1$. Thus, if both $x < 1$ and $x < 1/y$, then $H(\eta)$ is an analytic function, and $G_1(\eta)$ is a meromorphic function in the domain $x < 1$ and $x < 1/y$ (it is the ratio of two holomorphic functions). Here, its singularities are given by the zeros of $H(\eta)$.

be obtained from equation (2.5). Since the point $(y, x) = (1, 1)$ is ruled out by the above, it appears that the tricritical point is at $(y_c, x_c(y_c)) = ((1+\eta)^2/(1-\eta)^2, (1-\eta)^2/(1+\eta)^2)$.

If it is shown that $G_1(x, y, \eta)$ is analytic for all $0 \leqslant q < 1$ and for all $y > y_c$, and has essential singularities along $q = 1$, then this can be interpreted as evidence that $(y_c, x_c(y_c))$ is indeed the tricritical point. Proceed as follows. Divide equation (2.9) by $H(q\eta)$ to obtain

$$\alpha_2 \frac{H(q^2\eta)}{H(q\eta)} + (\alpha_1 + \eta) + \alpha_0 \frac{H(\eta)}{H(q\eta)} = 0. \tag{2.26}$$

Define $g(\eta) = H(q\eta)/H(\eta)$, so that

$$G_1(x, y, \eta) = \frac{1}{yq\eta}(g(\eta) - 1 - q\eta). \tag{2.27}$$

That the curve $q = 1$ is a locus of essential singularities in $G_1(x, y, \eta)$ can be argued as follows. Both $H(\eta)$ and $H(q\eta)$ have $q = 1$ as an accumulation point of poles. If these poles do not cancel, then they accumulate on $q = 1$. If they do cancel in the ratio $g(\eta) = H(q\eta)/H(\eta)$ (it can be shown that they do indeed cancel), then $g(\eta)$ is analytic at the points $q = y^{1/n}$ for $n = 1, 2, \ldots$. Observe that $H(\eta)$ has a root (a zero) between pairs of adjacent poles (that is, between $q = y^{1/n}$ and $q = y^{1/(n+1)}$ for $n = 1, 2, 3, \ldots$). At these roots, $g(\eta)$ has an infinity (since $H(q\eta) \neq H(\eta)$ for $q < 1$), and since the roots of $H(\eta)$ accumulate on $q = 1$ as well, the singularities in $g(\eta)$ accumulate on $q = 1$. Thus, $q = 1$ is an essential singularity in $G_1(x, y, \eta)$. To demonstrate analyticity of $G_1(x, y, \eta)$ for $q < 1$ and $y > y_c$, substitute $g(\eta)$ in equation (2.26) to obtain

$$g(\eta) = \frac{-\alpha_0}{\alpha_1 + \eta + \alpha_2 g(q\eta)}. \tag{2.28}$$

By replacing α_0, α_1 and α_2 with the expressions in equation (2.8) (and with $B = 1/yq$), and defining

$$\epsilon_p = 1 + yq - (y-1)q^{2+p}\eta \tag{2.29}$$

equation (2.28) simplifies to

$$g(\eta) = \frac{yq}{\epsilon_0(1 - \frac{1}{\epsilon_0} g(q\eta))}. \tag{2.30}$$

This can be developed into a continued fraction:

$$g(\eta) = \cfrac{yq}{\epsilon_0\left(1 - \cfrac{yq}{\epsilon_0\epsilon_1\left(1 - \cfrac{yq}{\epsilon_1\epsilon_2\left(1 - \cfrac{yq}{\epsilon_2\epsilon_3(1 - \cdots)}\right)}\right)}\right)} \tag{2.31}$$

and this also gives a continued fraction representation for $G_1(x, y, \eta)$. By Worpitsky's theorem (Wall 1967), this is convergent if

$$\sup_{p \geqslant 0} \left| \frac{yq}{\epsilon_p \epsilon_{p+1}} \right| \leqslant \tfrac{1}{4}. \tag{2.32}$$

Substituting ϵ_p gives explicitly

$$\sup_{p \geqslant 0} \left| \frac{yq}{(1 + yq - (y-1)q^{p+2}\eta)(1 + yq - (y-1)q^{p+3}\eta)} \right| \leqslant \tfrac{1}{4}. \tag{2.33}$$

If $y > 1$ and $q \leqslant 1$ then this implies that $G_1(x, y, \eta)$ is convergent for all values of y which satisfies

$$(1 + y - (y - 1)\eta)^2 \geqslant 4y \tag{2.34}$$

and solving for y it is found that (since $y > 1$),

$$y \geqslant \left(\frac{1+\eta}{1-\eta}\right)^2 = y_c. \tag{2.35}$$

In other words, $x_c(y) = 1/y$ if $y \geqslant y_c$.

If it is assumed that the curve of simple poles meets $xy = 1$ at $y = y_c$, then it follows from equation (1.6) that the limiting free energy in this model is

$$\mathcal{F}(y) \begin{cases} > \log y & \text{if} \quad y < y_c \\ = \log y & \text{if} \quad y > y_c. \end{cases} \tag{2.36}$$

Moreover, $\mathcal{F}(0) = 0$ and $\mathcal{F}(1) = \log(1 + \eta)$. The meeting of a line of poles with a curve of essential singularities at (y_c, x_c) is usually interpreted as a tricritical point in the phase diagram, and the asymptotic behaviour of $G_1(x, y)$ should be of the general form in equation (1.8). Much work has been done on the asymptotic behaviour of q-deformed factorials and Bessel functions (Prellberg 1995). In particular, the area–perimeter-generating function of staircase polygons (with x generating vertical edges, y generating horizontal edges and q generating area), has been determined to be (Brak and Guttmann 1990, Prellberg and Brak 1995)

$$G_s(x, y, q) = y\left(\frac{J(q^2x, qy, q)}{J(qx, qy, q)} - 1\right). \tag{2.37}$$

This generating function is known to have the following *uniform* asymptotic behaviour (Prellberg 1995). For $0 < x, y < 1$, $0 < q < 1$, define

$$z_m = (1 + y - x)/2 \qquad d = z_m^2 - y \tag{2.38}$$

and

$$\begin{aligned} 4\alpha^{3/2}/3 = {} & \log(z_m + \sqrt{d}) \log(1 - z_m + \sqrt{d}) - \log(z_m - \sqrt{d}) \log(1 - z_m - \sqrt{d}) \\ & + \mathcal{L}i_2(z_m - \sqrt{d}) + \mathcal{L}i_2(1 - z_m - \sqrt{d}) + \mathcal{L}i_2(z_m + \sqrt{d}) + \mathcal{L}i_2(1 - z_m + \sqrt{d}). \end{aligned} \tag{2.39}$$

Then

$$\begin{aligned} G_s(x, y, q) = {} & \frac{1}{2}\Bigg(1 - x - y + \alpha^{-1/2}(-\log q)^{1/3} \frac{\mathrm{Ai}'\left(\alpha(-\log q)^{-2/3}\right)}{\mathrm{Ai}\left(\alpha(-\log q)^{-2/3}\right)} \\ & \times \sqrt{(1 - x - y)^2 - 4xy}\Bigg)(1 + \mathrm{O}(-\log q)) \end{aligned} \tag{2.40}$$

where $\mathrm{Ai}(x)$ is an Airy function. In the case that $x = y$ in this model, it can be derived that the tricritical point is at $(q_c, x_c) = (1, \frac{1}{4})$, and the asymptotic behaviour in its vicinity is given by

$$G_s(x, x, q) \sim \tfrac{1}{2} - x + 4^{-2/3}(-\log q)^{1/3} \frac{\mathrm{Ai}'\left(4^{4/3}(\frac{1}{4} - x)(-\log q)^{-2/3}\right)}{\mathrm{Ai}\left(4^{4/3}(\frac{1}{4} - x)(-\log q)^{-2/3}\right)}. \tag{2.41}$$

A comparison with equation (1.8) shows that the tricritical exponents are

$$\phi = \tfrac{2}{3} \qquad 2 - \alpha_t = \tfrac{1}{3} \qquad 2 - \alpha_u = \tfrac{1}{2}. \tag{2.42}$$

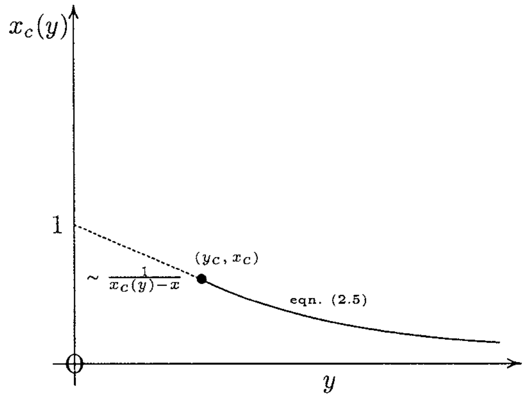

Figure 4. The expected singularity diagram of collapsing columns. A curve of singularities (presumably simple poles) in the generating function $G_1(x, y, \eta)$ meets a curve of essential singularities at the tricritical point denoted by •. The generating function is known to diverge as $1/(x_c(y) - x)$ at least at some points along the broken curve, but is finite along the curve of essential singularities, its value at each point is given by the expression in equation (2.5).

Note, furthermore, that the line $q = 1$ and $x \leqslant x_c$ is a line of essential singularities in $G_s(x, x, q)$, which changes into a curve of simpler singularities at the tricritical point.

These results are directly applicable to $G_1(x, y, \eta)$. In particular, put

$$X = (1 - 1/y)\eta \qquad Y = 1/yq \tag{2.43}$$

to obtain

$$G_1(x, y, \eta) = \frac{Y}{\eta}\left(\frac{J(Xq^2, Yq, q)}{J(Xq, Yq, q)} - 1\right) - qY. \tag{2.44}$$

With these definitions, the same type of asymptotic behaviour as in equation (2.40) is obtained. Note that $q = xy = 1$ is again a curve of essential singularities. Moreover, along $xy = 1$, the square-root factor in equation (2.40) becomes

$$\sqrt{(1 - X - Y)^2 - 4XY} = \sqrt{((1 - 1/y)^2(1 - \eta)^2 - 4\eta(1 - 1/y)/y} \tag{2.45}$$

and this vanishes at the points $y = 1$ and $y = (1 + \eta)^2/(1 - \eta)^2$. This last point is of course the presumed tricritical point in our model. Comparison with equation (1.10) indicates that $2 - \alpha_u = \frac{1}{2}$. Naturally, it is found that $\phi = \frac{2}{3}$, and so $2 - \alpha_t = \frac{1}{3}$. With the necessary reinterpretations of X, Y and η in equation (2.44) the asymptotic behaviour of histogram polygons (bar-graph polygons) is similarly obtained, with the same set of tricritical exponents.

3. Adsorbing columns

In this section a model of an adsorbing collection of connected columns is considered. In particular, visits (see figure 1) are allowed to occur, and an activity z conjugate to visits is introduced into the model. The generating function of this model is not difficult to obtain. Note that each collection of columns either has no visits (these have a generating function $G_1(x, y, \eta)$), or (a) is a visit or has a first column which is a visit, or (b) has a last column which is a visit, or (c) has a first visit which is neither the first nor last column. If $G(x, y, z, \eta)$

is the generating function of adsorbing columns, then these arguments show that

$$G(x, y, z, \eta) = z\eta + G_1(x, y, \eta) + z\eta G(x, y, z, \eta) + G_1(x, y, \eta)z\eta \\ +G_1(x, y, \eta)\, z\eta G(x, y, z, \eta). \tag{3.1}$$

Solving for $G(x, y, z, \eta)$ gives the generating function in terms of $G_1(x, y, \eta)$:

$$G(x, y, z, \eta) = \frac{z\eta + (1 + z\eta)G_1(x, y, \eta)}{1 - z\eta(1 + G_1(x, y, \eta))}. \tag{3.2}$$

Naturally, $G_1(x, y, \eta)$ is given by equations (2.21) or (2.27). The radius of convergence $x_c(y, z)$ of $G(x, y, z, \eta)$ again determines the limiting free energy in this model. An examination of equation (3.2) shows that $x_c(y, z)$ is determined by both $G_1(x, y, \eta)$ and by a simple pole when the denominator in equation (3.2) vanishes. The pole corresponds, in particular, to the adsorption transition. On the other hand, the singularity in $G_1(x, y, \eta)$ corresponds to the deflated–inflated transition, and as long as $1 - z\eta(1 + G_1(x, y, \eta)) > 0$, the radius of convergence of $G(x, y, z, \eta)$ is, in fact, determined by $G_1(x, y, \eta)$. The critical value of z is given by

$$z = \frac{1}{\eta(1 + G_1(x, y, \eta))} \tag{3.3}$$

for any given values of x and z. Of greater importance though is the radius of convergence of $G(x, y, z, \eta)$ for (fixed) given values of y and z, since this determines the phase behaviour in the model as defined by the free energy defined in equations (1.2) and (1.6).

Consider first $y < y_c$. In this case we look for a solution of x in equation (3.3) (which must be positive and finite). Observe that with increasing x, $G_1(x, y, \eta)$ approaches a (presumed) pole and increases without bound, so that a solution to (3.3) with $x \geqslant 0$ should exist provided that $z < 1/\eta$. Consider, for example, the special case that $y = 0$; in that event $G_1(x, 0, \eta) = x\eta/(1 - x)$ so that the solution of

$$z = \frac{1 - x}{\eta(1 - x + x\eta)} \tag{3.4}$$

determines the radius of convergence of $G(x, 0, z, \eta)$. The result is that

$$x_c(0, z) = \frac{1 - z\eta}{1 - z\eta + z\eta^2}. \tag{3.5}$$

This is positive for all $z < 1/\eta$. If $y = 1$ a similar argument, using equation (2.25), shows likewise that $x_c(1, z) > 0$ for all $z < 1/\eta$. In other words, as long as $z < 1/\eta$ and $y \leqslant y_c$, $G(x, y, z, \eta)$ has a pole at a critical value of $x = x_c(y, z) > 0$. If $z > 1/\eta$, then there are no solutions for x in equation (3.3), and the radius of convergence of $G(x, y, z, \eta)$ is zero. Thus, there is a phase boundary in the (z, y)-plane at $z = 1/\eta$ with $0 \leqslant y \leqslant y_c$. For values of $z < 1/\eta$ a deflated–inflated phase is obtained. At first glance it may seem that this phase should not be adsorbed, but since the singularity in equation (3.2) is determined by the fact that the denominator vanishes, and *not* by a singularity in $G_1(x, y, \eta)$, this should be a phase with a density of visits and thus an adsorbed phase. For values of $z \geqslant 1/\eta$ the model is clearly degenerate, corresponding to a model of columns with each column a visit (this immediately follows since $x_c(y, z) = 0$ in this regime, so that $G(0, y, z) = 0$ and thus $G(x, y, z, \eta) = z\eta/(1 - z\eta)$). These phases are indicated in figure 5 for $y < y_c$. Both phases are adsorbed, one corresponding to a model of columns in the deflated phase with a density of visits; the other to a model where every column is a visit, and the radius of convergence of $G(x, y, z, \eta)$ is zero.

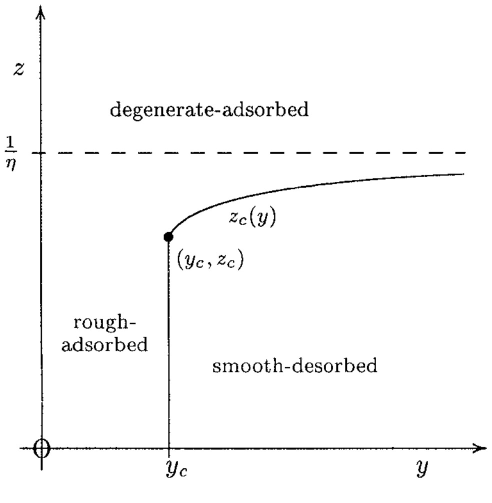

Figure 5. The phase diagram of collapsing and adsorbing columns. A multicritical point where the model is critical with respect to both the adsorbed–desorbed transition and the deflation–inflation transition is marked by •. This is often called a 'special point'.

Next, consider the situation that $y \geqslant y_c$. It is again the case that the radius of convergence of $G_1(x, y, z, \eta)$ is zero if $z > 1/\eta$; this follows from the same argument made in the last paragraph. Thus, assume that $z < 1/\eta$. The generating function $G_1(x, y, \eta)$ in equation (3.2) has an essential singularity if $x = x_c(y) = 1/y$ (and then its value is given by equation (2.5)). Since $G_1(x, y, \eta)$ is increasing with x, the radius of convergence of $G(x, y, z, \eta)$ will be determined by the radius of convergence of $G_1(x, y, \eta)$ provided that z is not too large. This occurs, in fact, for all z less than $z_c(y)$ given by

$$z_c(y) = \frac{2}{(1+\eta)(1-1/y) - \sqrt{(1-\eta-(1+\eta)/y)^2 - 4\eta^2/y}} \qquad \text{if} \quad y \geqslant y_c. \tag{3.6}$$

This expression is obtained by replacing $G_1(x, y, \eta)$ in equation (3.3) with equation (2.5) (and replacing x by $1/y$). Thus, if $z < z_c(y)$ and $y > y_c$, the phase is a smooth phase where the columns are also desorbed. If $z > z_c(y)$ (but less than $1/\eta$), then the radius of convergence of $G(x, y, z, \eta)$ is determined by the solution of equation (3.3), at which value of x a pole appears in $G(x, y, z, \eta)$. This is manisfestly an adsorbed phase, with a density of visits corresponding to the appearance of a simple pole. Notice that $z_c(y_c) = (1+\eta)/2\eta$, and that $z_c(y)$ is asymptotic to $1/\eta$ with increasing y. It is therefore apparent that the curve $z = z_c(y)$ determines a phase boundary consisting of adsorption transitions for all values of $y \geqslant y_c$.

On the other hand, the line $y = y_c$ and $0 \leqslant z \leqslant (1+\eta)/2\eta$ is a phase boundary separating a deflated phase from an inflated phase. In fact, if $z = 1$, then the model discussed in section 2 is recovered, and it appears that all the transitions along this phase boundary is in that class. The phases determined by $y < y_c$, and by $z_c(y) < z < 1/\eta$ and $y \geqslant y_c$, are, in fact, not separated by a phase boundary. These are both due to a simple pole in $G(x, y, z, \eta)$ determined by the solution (for x) of

$$1 - z\eta(1 + G_1(x, y, \eta)) = 0. \tag{3.7}$$

It follows from the implicit function theorem that $x_c(y, z)$ is analytic, and so there is no phase boundary separating the phases for $y < y_c$, $z < 1/\eta$ and $y \geqslant y_c$ with $z_c(y) < z < 1/\eta$. The point $(y, z) = ((1+\eta)^2/(1-\eta)^2, (1+\eta)/2\eta)$ is a multicritical point where phase boundaries corresponding to adsorption transitions, and inflation–deflation transitions meet. Thus, the model is critical at this point both with respect to adsorption and inflation, such points are called special points in the literature (Vanderzande 1998).

The scaling exponents associated with the adsorption transition at $y = y_c$ can be obtained from equation (3.6). In particular, the critical value of x is $x_c = (1-\eta)^2/(1+\eta)^2 = 1/y_c$ and solving for $x_c - x$ in terms of $z_c - z$ explicitly (at this value of y) gives

$$x_c - x = \frac{16\eta^3(z_c - z)^2}{(1+\eta)^2((1+\eta)^2 - 4\eta^2(z_c - z)^2)}. \tag{3.8}$$

The classical theory of tricritical scaling proposes that the scaling axes at the adsorption critical point should be $g = x_c - x$ and $t = z - z_c$, and moreover, the shift exponent ψ is defined by $g \sim t^{\psi}$. Comparison with equation (3.8) shows that $\psi = 2$. The crossover exponent is related to ψ by $\psi = 1/\phi$, and thus $\phi = \frac{1}{2}$. Note that $G_1(x, y, \eta)$ is finite at the multicritical point, and thus the divergence in $G(x, y, z, \eta)$ is a simple pole here, as z approaches its critical value. This indicates that $\gamma_u = 1$, and from equation (1.7) we conclude that $\gamma_t = \frac{1}{2}$. The values of these exponents are consistent with the classical critical exponents associated with linear polymer adsorption (for example, they were also obtained in a directed-walk model of polymer adsorption (Janse van Rensburg 1999), and in a model of adsorbing partially directed walks (Whittington 1998)).

4. Conclusions

In this paper I considered a model of self-interacting columns which may undergo a deflated–inflated transition or a desorbed–adsorbed transition. The generating function of the model was derived using a functional recursion, and the limiting free energy $\mathcal{F}(y)$ in an ensemble with only a contact activity was shown to be a non-analytic function. The generating function, and the techniques I used to study it, are related to other results in this area. In particular, the generating function is related to that of staircase polygons (Brak and Guttmann 1990), and its phase diagram can be shown to contain the phase diagram of histogram polygons as a special case. Histogram polygons, in turn, have an upper perimeter which is a partially directed walk. Thus, the inflation–deflation transition in the model of interacting columns can be associated with transitions in all of these models. In particular, the phase diagram, and the arguments in section 2.2 (as well as, in particular, equations (2.28) and (2.31)) have very similar structures to results obtained for a collapsing partially directed walk (Brak *et al* 1992, see, in particular, equations (4.5)–(4.7) in that paper), and it is unsurprising that the same critical exponents for collapsing partially directed walks and self-interacting collections of columns are found.

In section 3 the contact-visit generating function of interacting columns was found. In addition, the phase diagram of this model was argued to be given by figure 5. Some parts of this are now exactly known. For example, the shape of the critical curve is given by equation (3.6) for $y > y_c$, where y_c is the critical contact activity. For $y < y_c$ the situation is not exact, but relies on educated guesses. We noted that for $y < 1$ there are simple poles in $G_1(x, y, \eta)$ at values of $x_c(y)$ less than $1/y$. If it could be determined that these do, in fact, define the radius of convergence of $G_1(x, y, \eta)$ for all $y < y_c$, then the phase diagram as illustrated in figure 5 is correct.

The tricritical scaling form of the inflation–deflation transition can be read from the (known) asymptotic form of inflating staircase polygons (Prellberg 1995). The (tricritical) scaling exponents found in this model are also the same as for inflating staircase polygons or histogram polygons (Brak and Guttmann 1990) or collapsing partially directed walks (Brak *et al* 1992; see also Janse van Rensburg 2000). The critical exponents associated with the adsorption transition at the critical contact activity was also found to be those obtained in other studies of adsorbing directed and partially directed walks (Batchelor and Yung 1995, Whittington 1998, Janse van Rensburg 1999). An outstanding issue is the asymptotic form of the generating function $G(x, y, z, \eta)$ in equation (3.2), although some information on this can be obtained by considering equations (2.40) and (2.43).

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